

NOTE

On the balanced decomposition number

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Abstract

A *balanced coloring* of a graph G means a triple $\{P_1, P_2, X\}$ of mutually disjoint subsets of the vertex-set $V(G)$ such that $V(G) = P_1 \uplus P_2 \uplus X$ and $|P_1| = |P_2|$. A *balanced decomposition* associated with the balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of G is defined as a partition of $V(G) = V_1 \uplus \cdots \uplus V_r$ (for some r) such that, for every $i \in \{1, \dots, r\}$, the subgraph $G[V_i]$ of G is connected and $|V_i \cap P_1| = |V_i \cap P_2|$. Then the *balanced decomposition number* of a graph G is defined as the minimum integer s such that, for every balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of G , there exists a balanced decomposition $V(G) = V_1 \uplus \cdots \uplus V_r$ whose every element $V_i (i = 1, \dots, r)$ has at most s vertices. S. Fujita and H. Liu [SIAM J. Discrete Math. 24, (2010), pp. 1597–1616] proved a nice theorem which states that the balanced decomposition number of a graph G is at most 3 if and only if G is $\lfloor \frac{|V(G)|}{2} \rfloor$ -connected. Unfortunately, their proof is considerably long (about 10 pages) and complicated. Here we give an immediate proof of the theorem.

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1 Introduction

Throughout this paper, we only consider finite undirected graphs with no multiple edges or loops. For a graph G , let $V(G)$ and $E(G)$ denote the vertex-set of G and the edge-set of G , respectively. For a subset $X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced by X , and $N_G(X)$ denotes the set $\{y \in V(G) \setminus X \mid \exists x \in X, \{x, y\} \in E(G)\}$. This set $N_G(X)$ is called the *open neighborhood* of X in G . A subset $Y \subseteq V(G)$ is called a *vertex-cut* of G if there is a partition $V(G) \setminus Y = X_1 \uplus X_2$ such that $|X_i| \geq 1$ and $N_{G[V(G) \setminus Y]}(X_i) = \emptyset$ ($i = 1, 2$). For other basic definitions in graph theory, please consult [2].

In 2008, S. Fujita and T. Nakamigawa [4] introduced a new graph invariant, namely the *balanced decomposition number* of a graph, which was motivated by the estimation of the number of steps for pebble motion on graphs. A *balanced coloring* of a graph G means a triple $\{P_1, P_2, X\}$ of mutually disjoint subsets of $V(G)$ such that $V(G) = P_1 \uplus P_2 \uplus X$ and $|P_1| = |P_2|$. Then a *balanced decomposition* of G associated with its balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ is defined as a partition of $V(G) = V_1 \uplus \cdots \uplus V_r$ (for some r) such that, for every $i \in \{1, \dots, r\}$, $G[V_i]$ is connected and $|V_i \cap P_1| = |V_i \cap P_2|$. Note that every disconnected graph has a balanced coloring which admits no balanced decompositions. Now the *balanced decomposition number* of a connected graph G is defined as the minimum integer s such that, for every balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of G , there exists a balanced decomposition $V(G) = V_1 \uplus \cdots \uplus V_r$ whose every element V_i ($i = 1, \dots, r$) has at most s vertices.

The set of the starting and the target arrangements of mutually indistinguishable pebbles on a graph G can be modeled as a balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of G . Then, as is pointed out in [4], the balanced decom-

position number of G gives us an upper-bound for the minimum number of necessary steps to the pebble motion problem, and, for several graph-classes, this upper bound is sharp.

In addition to the initial motivations and their applications in [4], this newcomer graph invariant turns out to have deep connections to some essential graph theoretical concepts. For example, the following conjecture in [4] indicates a relationship between this invariant and the vertex-connectivity of graphs:

Conjecture 1. (S. Fujita and T. Nakamigawa (2008)) *The balanced decomposition number of G is at most $\lfloor \frac{|V(G)|}{2} \rfloor + 1$ if G is 2-connected.*

Recently, G. J. Chang and N. Narayanan [1] announced a solution to this conjecture.

Then especially, S. Fujita and H. Liu [3] proved the affirmation of the “high”-connectivity counterpart of the above conjecture, as follows:

Theorem 1. (S. Fujita and H. Liu (2010)) *Let G be a connected graph with at least 3 vertices. Then the balanced decomposition number of G is at most 3 if and only if G is $\lfloor \frac{|V(G)|}{2} \rfloor$ -connected.*

Thus, there may be a trade-off between the vertex-connectivity and the balanced decomposition number. This interesting relationship should be investigated for its own sake.

Their proof of Theorem 1 in [3] as well as the statement of the theorem has some importance, because it indicates a deep relationship between this new invariant and graph matching. Indeed, the proof heavily uses the typical techniques in classical matching algorithms. The concept of the balanced decomposition can be regarded as a generalization of graph matching, and their proof suggests that there exists a similarity strictly more than its appearance.

2 A quick proof of Theorem 1

The disadvantage of the proof of Theorem 1 in [3] is its length (about 10 pages) and complicatedness. We give a shorter proof here.

Proof of Theorem 1. In order to prove the **if part**, let us define the following new bipartite graph H from a given balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of a graph G :

1. The partite sets of H are $V_1(H) := P_1 \uplus X_1$ and $V_2(H) := P_2 \uplus X_2$, where each $X_i := \{(x, i) \mid x \in X\}$ ($i = 1, 2$) is a copy of the set $X (\subseteq V(G))$.
2. The edge set $E(H)$ of H is defined as follows:

$$\begin{aligned} E(H) := & \{ \{p_1, p_2\} \mid p_1 \in P_1, p_2 \in P_2, \{p_1, p_2\} \in E(G) \} \\ & \cup \{ \{p_1, (x, 2)\} \mid p_1 \in P_1, x \in X, \{p_1, x\} \in E(G) \} \\ & \cup \{ \{(x, 1), p_2\} \mid x \in X, p_2 \in P_2, \{x, p_2\} \in E(G) \} \\ & \cup \{ \{(x, 1), (x, 2)\} \mid x \in X \}. \end{aligned}$$

Then clearly, the balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of G has a balanced decomposition $V(G) = V_1 \uplus \dots \uplus V_r$ whose every element $V_i (i = 1, \dots, r)$ consists of at most 3 vertices, if and only if the graph H has a perfect matching. Then we use here the famous ‘‘Hall’s Marriage Theorem’’ [5], as follows.

Lemma 2. (P. Hall(1935)) *Let G be a bipartite graph whose partite sets are $V_1(G)$ and $V_2(G)$. Suppose that $|V_1(G)| = |V_2(G)|$. Then G has a perfect matching if and only if every subset U of $V_1(G)$ satisfies $|U| \leq |N_G(U)|$.*

Now, suppose that H does not have any perfect matching. Then, from lemma 2, $\exists A \subseteq P_1, \exists B \subseteq X_1, |N_H(A \cup B)| \leq |A| + |B| - 1$. Let $C := P_2 \setminus N_H(A \cup B)$ and $D := X_2 \setminus N_H(A \cup B)$. Then, by symmetry, $|N_H(C \cup D)| \leq$

$|C| + |D| - 1$ also holds. Furthermore, by the definition of H , $|B| \leq |X_2 \setminus D|$ and $|D| \leq |X_1 \setminus B|$ hold, and hence $0 \leq |X| - |B| - |D| \leq |A| + |C| - |P_1| - 1 = |A| + |C| - |P_2| - 1$ satisfies. Please see Figure 1 which shows this situation. The vertex-cut of $V(G)$ corresponding to the set $(P_1 \setminus A) \cup (P_2 \setminus C) \cup (X_1 \setminus B)$

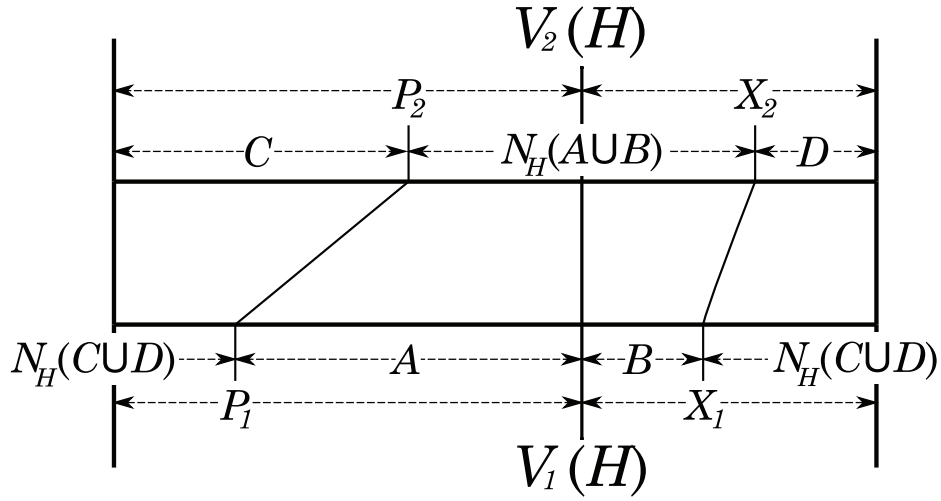


Figure 1: The bipartite graph H which has no perfect matching.

separates $G[C]$ from its remainder. By symmetry, the vertex-cut of $V(G)$ corresponding to the set $(P_1 \setminus A) \cup (P_2 \setminus C) \cup (X_2 \setminus D)$ separates $G[A]$ from its remainder. Hence if G is $\lfloor \frac{|V(G)|}{2} \rfloor$ -connected, $|V(G)| - 1 \leq 2(|P_1| - |A| + |P_2| - |C|) + (|X| - |B|) + (|X| - |D|) = (|P_1| + |P_2| + |X|) - 2(|A| + |C| - |P_1|) - (|X| - |B| - |D|) \leq |V(G)| - 2$, a contradiction.

The proof of the **only if part** is given by a construction of special balanced colorings, which is the same as the original one in [3]. We will transcribe the construction only for the convenience of readers.

Suppose that G is not $\lfloor \frac{|V(G)|}{2} \rfloor$ -connected. And let Y denote a minimum vertex-cut of G . Note that $2|Y| \leq |V(G)| - 2$. Then $G[V(G) \setminus Y]$ is divided into two graphs G_1 and G_2 such that $|V(G_i)| \geq 1$ and $N_{G[V(G) \setminus Y]}(V(G_i)) = \emptyset$ ($i = 1, 2$). Without loss of generality, we assume that $|V(G_1)| \leq |V(G_2)|$. Let l denote the number $\min\{|Y|, |V(G_1)| - 1\}$. Suppose an arbitrary balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of G such that $|Y \cap P_1| = l$ and $|Y \cap P_2| = |Y| - l$ and $|V(G_1) \cap P_2| = l + 1$ and $V(G_1) \cap P_1 = \emptyset$. Then, it is easy to see that every balanced decomposition associated with such a balanced coloring has at least one component whose vertex-size is at least 4, that is, the balanced decomposition number of G is at least 4. ■

References

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